Ex3: $X = X_1 \cup \dots \cup X_r$ puredim, Since $1 = \deg X \underset{\uparrow}{=} \sum_{i=1}^{r} \deg X_i \Rightarrow r = 1$

$\Rightarrow X$ is irreducible

Rk 7.15

(1) Proof of ~~this~~ the hint:

$H \overset{=V(f)}{}$ ~~hypersurface~~ linear plane and H contains 2-points

If $X \not\subset H$, then by Bézout's theorem:

$$\# |X \cap H| \leq \deg X \cdot \deg f = 1 \quad \text{(Corollary 7.19 (a))}$$

but we have at least 2 points

Hence, we can not apply Bézout's th. $\Rightarrow X \subset H$.

Assume X is not a line $\Rightarrow \exists\ A, B, C$ ~~di~~ $\in X$ different, such that $C \notin AB$.

(corresponding vektors $\vec{A}, \vec{B}, \vec{C}$ are lin. independent)

Then by LA $\exists$ a hyperplane H containing A and B, but not C

$$A, B \in H \underset{\text{Hint}}{\Longrightarrow} X \subset H \Rightarrow C \in X \subset H \quad \lightning$$

It follows $X =$ line.

(2) Induction on $\dim X = r$

~~Same~~ $\deg X = 1$ + X pure-dim. $\underset{\uparrow}{\Rightarrow}$ X irreducible

same arg. as in (1)

Any linear subspace has $\deg = 1$ (by Example 7.14)

Assume $\dim X = r \geq 2$, $\deg X = 1$

Take a H hyperplane with $X \not\subset H = V(f)$

By Bézout's Theorem $\underset{\deg Y = \deg I(Y)}{}$ $\leq \deg(I(X) + f) = \deg X \cdot \deg f = 1$

~~$\forall$~~ ~~X~~ $Y = V(I(X) + f) = X \cap V(f)$

$$I(Y) = \sqrt{I(X) + f} \supseteq I(X) + f$$

It follows, ~~$\deg I(Y) = 1$~~ $\deg Y = 1$ and $\dim Y = \dim X - 1 = r - 1$

Moreover, Y is pure-dim. by Ex2, so By question (1):

Y is a linear subspace, $Y = \mathbb{P}(U)$, $\dim U = r$

(*) Assume a hyperplane H contains $A_1, \dots, A_{r+1} \in X$, s.t. $\dim \langle \vec{A_1}, \dots, \vec{A_{r+1}} \rangle = r+1$ linearly independent

Then $X \subset H$ (otherwise $\lightning$ by Bézout's theorem)

If $X \neq \mathbb{P}(U)$, with $\dim U = r+1$

Then $\exists$ $r+2$ points $A_1, A_2, \dots, A_{r+1}, A_{r+2}$

s.t. $\dim \langle \vec{A}_1, \dots, \vec{A}_{r+1} \rangle = r+1$ and $\vec{A}_{r+2} \notin \langle \vec{A}_1, \dots, \vec{A}_{r+1} \rangle$

$\exists$ a hyperplane H, $\vec{A}_1, \dots, \vec{A}_{r+1} \in H$ and $\vec{A}_{r+2} \notin H$

By $(*)$ $X \subset H$, but $A_{r+2} \in X$ and $A_{r+2} \notin H$ $\lightning$

So X is a $\mathbb{P}(\mathcal{U})$, with $\mathcal{U} = \langle \vec{A}_1, \dots, \vec{A}_{r+1} \rangle$, $\dim \mathcal{U} = r+1$

Ex 4

$X_1 = AE \cup BF \cup DC$ both cubics.

$X_2 = BD \cup CE \cup AF$

$X_i = V(f_i)$, f_i = product of 3 lin. polyn

$\deg f_i = 3$

$\deg X_i = 3$

Note that by Bézout's Th.

$\ell_1 \cap X_i = \{A, B, C\}$

$\ell_2 \cap X_i = \{D, E, F\}$

Hence $f_i(S) \neq 0$

Choose, $\alpha, \beta \in K$ s.t. $(\alpha f_1 + \beta f_2)(S) = 0$

$\alpha, \beta \neq 0$

$\alpha f_1 + \beta f_2 \neq 0$ otherwise $f_1 = \gamma f_2$ and

$X_1 = V(f_1) = V(f_2) = X_2$ $\lightning$

$*$ $\tilde{f} = \alpha f_1 + \beta f_2$, $\tilde{X} = V(\tilde{f})$ $\deg \tilde{X} = 3$

$\tilde{X} \cap \ell_1 = \{S, A, B, C\}$ $\xRightarrow{\text{Bézout theorem}}$ ℓ_1 is a component of $\tilde{X}$

$\tilde{X} \cap \ell_2 = \{S, D, E, F\}$ $\Rightarrow$ ℓ_2 is a component of $\tilde{X}$

$\tilde{X} = \ell_1 \cup \ell_2 \cup \ell$

$Q, P, R \in X_1, X_2 \Rightarrow Q, P, R \in \tilde{X}$ but $\notin \ell_1, \ell_2 \Rightarrow Q, P, R \in \ell$.